

FURTHER REMARKS ON AN ORDER FOR QUANTUM OBSERVABLES

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ABSTRACT. S. Gudder and, later, S. Pulmanová and E. Vinceková, have studied in two recent papers a certain ordering of bounded self-adjoint operators on a Hilbert space. We present some further results on this ordering and show that some structure theorems of the ordered set of operators can be obtained in a more abstract setting of posets having the upper bound property and equipped with a certain orthogonality relation.

1. INTRODUCTION

In [13], S. Gudder introduced a certain order for quantum observables, in fact, on the set $\mathcal{S}(H)$ of bounded self-adjoint operators on a complex Hilbert space H , and suggested to call it the *logical order*. He demonstrated, in particular, that $\mathcal{S}(H)$ is a generalized orthoalgebra; the logical order is by definition the natural order of this algebra. He also showed that this is not a lattice order, but nevertheless every pair of observables having a common upper bound has a join and a meet with respect to this order. Such a poset was called by him a *near-lattice* (but see the beginning of the next section). Actually, every initial segment of $\mathcal{S}(H)$ is even a σ -orthomodular lattice. Also the commutative case (in which observables are represented by random variables on a probability space) was considered in [13]; as noted by the author of that paper, this case actually served as motivation and a source of intuition for results and proofs of the general case.

The properties of the new ordering were studied in more detail by S. Pulmannová and E. Vinceková in [19], where several results of [13] were essentially improved. These authors observed that the logical order is actually a restriction of Drazin's order (nowdays usually called star order or $*$ -order) introduced by him in [10] for all bounded operators on H . They proved, for example, that $\mathcal{S}(H)$ is even a weak generalized orthomodular poset. Moreover, this poset is *bounded complete*, i.e., every subset bounded from above has a join and, correspondingly, every nonempty subset has a meet.

More recently, existence conditions of joins and meets in $\mathcal{S}(H)$ (under the logical ordering), and representations of these operations have been discussed, e.g., in [12, 15, 16, 20, 21].

We present here some further results on the order structure of $\mathcal{S}(H)$, and also fill two small gaps in proofs in [13]. In particular, we obtain explicit descriptions for the Gudder join and meet operations in terms of operator composition and

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lattice operations on projectors, and a simple proof of bounded completeness of $\mathcal{S}(H)$. On the other hand, we discuss some of the properties of the poset $\mathcal{S}(H)$ in a more abstract setting, and show that the aforementioned properties of the logical order of observables are not quite independent. For instance, any poset having the least element and possessing the so called *upper bound property* (any pair of elements has a join if they have a common upper bound), if equipped with an appropriate orthogonality relation, carries a structure of a generalized orthoalgebra, in which every initial segment is an orthomodular lattice. Moreover, there is a non-commutative total binary operation (called *skew meet*) on $\mathcal{S}(H)$ which, considered together with the partial join operation, turns the poset into a so called *skew nearlattice*. This allows, on the one hand, to establish a link between structures arising in quantum logic and some branches of the theory of information systems (where the notion of skew nearlattice has emerged; see [5]), and on the other hand, to apply to the algebra $\mathcal{S}(H)$ certain general decomposition and structure theorems from [6]. We do not address, however, these questions in the present paper.

The structure of the paper is as follows. The subsequent section contains the necessary background on posets having the upper bound property, known also as nearsemilattices in algebra and as a (simple version of) domains in database theory. Three principal examples of such posets, including $\mathcal{S}(H)$, are considered in Section 3. Some order properties of $\mathcal{S}(H)$ are discussed also in Section 4. Section 5 deals with so called quasi-orthomodular nearsemilattices, which mimic, in a sense, the generalized orthomodular lattices of [14], and Section 6, with skew nearlattices.

2. PRELIMINARIES: NEARSEMI LATTICES

A *nearlattice* is usually defined as a meet semilattice having the upper bound property. Therefore, every bounded complete poset is an example of a nearlattice. Equivalently, a nearlattice is a meet semilattice in which every initial segment is a lattice. Since early eighties, such structures have been intensively studied by W. Cornish and his collaborators; see, e.g., [8, 9, 18]. Some authors prefer order duals of such algebras [2].

Arbitrary posets having the upper bound property were named (upper) *near-semilattices* in [3]. Thus, a nearlattice may be viewed also as a nearsemilattice that happens to be a meet semilattice. Near-lattices mentioned in [13] (see Introduction) is a weaker concept.

We shall always assume that a near(semi)lattice has the least element 0, and consider such structures as partial algebras of kind $(A, \vee, 0)$, resp., $(A, \wedge, \vee, 0)$, where \wedge is, as usual, the meet operation and \vee is the partial join operation. The following axiomatic description of nearsemilattices goes back to [3, Section 1] (for arbitrary near(semi)lattice terms s and t , we write $s \circ t$ to mean that $s \vee t$ is defined under that or other intended assignment of values to variables occurring in these terms).

Proposition 1. *An algebra $(A, \vee, 0)$, where 0 is a nullary operation and \vee is a partial binary operation, is a nearsemilattice if and only if it fulfils the conditions*

- (\vee_1): $x \circ x$ and $x \vee x = x$,
- (\vee_2): if $x \circ y$, then $y \circ x$ and $x \vee y = y \vee x$,
- (\vee_3): if $x \circ y$ and $x \vee y \circ z$, then $y \circ z$, $x \circ y \vee z$ and $(x \vee y) \vee z = x \vee (y \vee z)$,
- (\vee_4): $x \circ 0$ and $x \vee 0 = x$.

The order relation on a nearsemilattice A is recovered from the partial join operation as follows:

$$x \leq y \text{ if and only if } x \circ y \text{ and } x \vee y = y, \quad (1)$$

and then $x \vee y$ (when defined) is the join of x and y w.r.t. this order, while 0 is the least element.

Corollary 2. *An algebra $(A, \wedge, \vee, 0)$ is a nearlattice if and only if $(A, \vee, 0)$ is a nearsemilattice, (A, \wedge) is a semilattice, and the following absorption laws are fulfilled:*

$$\begin{aligned} (\vee_5): & \text{ if } x \circ y, \text{ then } x \wedge (x \vee y) = x, \\ (\vee_6): & \text{ } x \wedge y \circ y \text{ and } (x \wedge y) \vee y = y. \end{aligned}$$

Observe that these laws can also be rewritten in a \vee -free form, where \leq is the ordering (1):

$$\text{if } x \leq y, \text{ then } x \wedge y = x, \text{ and } x \wedge y \leq y. \quad (2)$$

A nearsemilattice A is said to be *distributive* [7], if every initial segment of it is a distributive semilattice:

$$(\vee_7): \text{ if } y \circ z \text{ and } x \leq y \vee z, \text{ then } x = y' \vee z' \text{ for some } y' \leq y \text{ and } z' \leq z.$$

In the case when A happens to be a nearlattice, this reduces to the standard notion of a distributive nearlattice (every initial segment is a distributive lattice).

A *De Morgan complementation*, or just *m-complementation*, on a poset is a unary operation $-$ such that

$$x^{--} = x, \text{ and if } x \leq y, \text{ then } y^- \leq x^-.$$

By a *sectionally m-complemented poset* we mean a poset with the least element, in which every initial segment $[0, p]$ is equipped with an m-complementation $\bar{}$ [7, Section 2]. In such a poset, a partial subtraction operation \ominus may be defined by

$$x \ominus y = z \text{ if and only if } y \leq x \text{ and } z = y_x^- \text{ in } [0, x].$$

Lemma 2.5 of [7], together with Proposition 2.2, Theorem 2.3 and Corollary 2.4 of that paper, leads us to the following characterization of certain nearlattices.

Proposition 3. *A sectionally m-complemented poset is a nearlattice if and only if the partial subtraction \ominus on it can be extended to a total operation $-$ satisfying conditions*

$$\begin{aligned} (-_1): & \text{ if } x \leq y, \text{ then } z - y \leq z - x, \\ (-_2): & \text{ } x - (x - y) \leq y, \\ (-_3): & \text{ } x - 0 = x. \end{aligned}$$

Namely, in an m-complemented nearlattice,

$$x - y = x \ominus (x \wedge y) \quad (3)$$

(see equation (7) in [7]). Explicit definitions of join and meet in terms of subtraction are not given in [7]. We omit the tedious calculations and note without proof that

$$x \wedge y = x - (x - y), \text{ and } x \vee y = z - ((z - x) \wedge (z - y)) \text{ whenever } x, y \leq z.$$

An ordered algebra $(A, -, 0)$, where 0 is the least element and $-$ is a binary operation satisfying $(-_1)$ – $(-_3)$, is a weak BCK-algebra in the sense of [7]; see Proposition 2.2 therein.

3. EXAMPLES

Let us now consider three examples of nearlattices. The first example may be considered as motivating the concept; the other two are borrowed from [13, 19].

Example 1 (partial functions). Let I and V be nonempty sets, and let $\mathcal{PF}(I, V)$ be the set of all partial functions from I to V . It is partially ordered by set inclusion, and $(\mathcal{PF}(I, V), \cap, \cup, \lambda)$, where λ is the nowhere defined function, is a nearlattice with respect to this order and usual operations \cap and \cup . Actually, $\mathcal{PF}(I, V)$ is even bounded complete, and total functions are just its maximal elements.

Observe that the union of two functions in $\mathcal{PF}(I, V)$ is defined if and only if they agree on the common part of their domains. It is easily seen that $\text{dom}(\varphi \cup \psi) = \text{dom } \varphi \cup \text{dom } \psi$ and $\text{dom}(\varphi \cap \psi) \subseteq \text{dom } \varphi \cap \text{dom } \psi$. Any nearlattice $(\Phi, \cup, \cap, \lambda)$, where Φ is a subset of some $\mathcal{PF}(I, V)$, is called a *functional nearlattice*. If the nearlattice contains all pairwise unions (of elements of Φ) existing in $\mathcal{PF}(I, V)$, it is said to be *closed*.

Every initial segment of a functional nearlattice Φ is even a Boolean lattice. Then the corresponding BCK-subtraction (3) coincides in Φ with set subtraction.

Example 2 (random variables). Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and let $\mathcal{M}(\mathcal{A})$ be the set of random variables on this space (i.e., measurable functions $\Omega \rightarrow \mathbb{R}$). Put $\text{supp } f := \{\omega \in \Omega : f(\omega) \neq 0\}$. For $f, g \in \mathcal{M}(\mathcal{A})$, write $f \perp g$ if $fg = 0$ (i.e., $\text{supp } f \cap \text{supp } g = \emptyset$), and put $f \preceq g$ if there is a function $h \in \mathcal{M}(\mathcal{A})$ such that $f \perp h$ and $f + h = g$. The relation \preceq is an order on $\mathcal{M}(\mathcal{A})$. By [13, Theorem 3.1], $f \preceq g$ iff $f(\omega) = g(\omega)$ whenever $\omega \in \text{supp } f$; equivalently, iff $f = g\chi_{\text{supp } f}$ (χ_K stands for the characteristic function of a subset $K \subseteq \Omega$). Direct calculations show that it is a nearlattice ordering with the zero function 0 as its least element (Theorem 3.5 in [13]). The corresponding meet and join operations, \wedge and \vee , may be defined as follows:

$$f \wedge g := g\chi_{\{\omega \in (\text{supp } f \cap \text{supp } g) : f(\omega) = g(\omega)\}} = g\chi_{\{\text{supp } f \cap \text{supp } g \setminus \text{supp}(f-g)\}}$$

and, if $f, g \preceq h$,

$$f \vee g := h\chi_{(\text{supp } f \cup \text{supp } g)}.$$

By the way, $f \wedge g = h\chi_{(\text{supp } f \cap \text{supp } g)}$ in this case. (We have changed the notation of [13], where these operations were denoted by \wedge and \vee , respectively.) Similarly, if F is a subset of $\mathcal{M}(\mathcal{A})$ with an upper bound h , then $h\chi_{(\bigcup \{\text{supp } f : f \in F\})}$ is the least upper bound of F : $\mathcal{M}(\mathcal{A})$ is bounded complete.

Observe that this example is essentially subsumed under the previous one: the nearlattice $\mathcal{M}(\mathcal{A})$ may be identified with the closed subalgebra $M_{\mathcal{A}}$ of $\mathcal{PF}(\Omega, \mathbb{R}_0)$ (where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$) obtained by replacing every function in $\mathcal{M}(\mathcal{A})$ with its codomain restriction to \mathbb{R}_0 ; this transfer is an isomorphism of $\mathcal{M}(\mathcal{A})$ into $\mathcal{PF}(\Omega, \mathbb{R}_0)$. In particular, every initial segment $[0, g]$ in $\mathcal{M}(\mathcal{A})$ is a Boolean lattice, where the complementation of f is $g - f$. Then the BCK-subtraction in $\mathcal{M}(\mathcal{A})$ (which we denote by $\dot{-}$) is given by

$$g \dot{-} f = g - (f \wedge g) = g\chi_{\text{supp}(f-g)}.$$

Example 3 (quantum observables). We return to the set $\mathcal{S}(H)$ of bounded self-adjoint operators on a Hilbert space. The ensuing conventions follow to [13]. Let $\mathcal{P}(H)$ stand for the set of all projections (idempotent operators). If $A \in \mathcal{S}(H)$, denote by $\text{ran } A$ the range of A and by $\overline{\text{ran } A}$, its closure. Denote the projection

onto $\overline{\text{ran}} A$ by P_A . For $A, B \in \mathcal{S}(H)$, put $A \perp B$ if the composition AB is the zero operator O (equivalently, if $\overline{\text{ran}} A \perp \overline{\text{ran}} B$, or $P_A P_B = O$; see Lemma 4.1 in [13]), and put $A \preceq B$ if $B = A + C$ for some $C \in \mathcal{S}(H)$ with $C \perp A$. By [13, Lemma 4.3], $A \preceq B$ if and only if $Ax = Bx$ whenever $x \in \overline{\text{ran}} A$ (then $\overline{\text{ran}} A \subseteq \overline{\text{ran}} B$), or, equivalently, if $A = BP_A$ (then B and P_A commute). The relation \preceq is the logical order on $\mathcal{S}(H)$ mentioned in Introduction.

We shall say that the closed subspace $\overline{\text{ran}} A$ of H *corresponds* to A . Recall that the transfer $P_A \mapsto \overline{\text{ran}} A$ from projection operators to closed subspaces is one-to-one and onto; moreover, $\mathcal{P}(H)$ is ordered by

$$P_1 \leq P_2 \text{ if and only if } P_1 P_2 = P_1 \text{ if and only if } P_1 = P_2 P_1,$$

and then $P_A \leq P_B$ if and only if $\overline{\text{ran}} A \subseteq \overline{\text{ran}} B$. In particular, $\mathcal{P}(H)$ is lattice ordered, and $P_A \leq P_B$ whenever $A \preceq B$. Corollary 4.4 in [13] implies that $P_A \leq P_B$ if and only if $P_A \preceq P_B$.

We denote the join and meet operations in the lattice $\mathcal{P}(H)$ by \vee and \wedge , and these operations in $\mathcal{S}(H)$ (which may be partial), by \vee and \wedge , respectively (in [13, 19], the same symbols \vee and \wedge are used for both purposes). According to [13, Corollary 4.13], meets and joins exist in $\mathcal{S}(H)$ for every pair of elements bounded above (but see the beginning of the next section). Using this result as the base, it is shown in [19, Corollary 4.7] that $\mathcal{S}(H)$ is even bounded complete, in particular, a nearlattice (evidently, O is the least element in it).

It is easily seen that, in every interval $[O, B]$ of $\mathcal{S}(H)$, $B - A$ is an m -complement of A . If the symbol $\dot{-}$ stands for the BCK-subtraction (3) in the nearlattice $\mathcal{S}(H)$, then (for arbitrary $A, B \in \mathcal{S}(H)$)

$$B \dot{-} A = B - (A \wedge B) = B - BP_{A \wedge B} = B(I - P_{A \wedge B}).$$

This example is not fully subsumed under Example 1. The definition of \preceq shows that $A = B$ if and only if $A|_{\overline{\text{ran}} A} = B|_{\overline{\text{ran}} B}$. Therefore, an operator in $\mathcal{S}(H)$ is completely determined by its restriction to the corresponding subspace of H . We thus can identify every operator A with the partial function $A|_{\overline{\text{ran}} A}$ from $\mathcal{PF}(H, H)$, and come in this way to a nearlattice of functions S_H (ordered by set inclusion) isomorphic to $\mathcal{S}(H)$. However, as the join of two subspaces generally differs from their set-theoretical union, this nearlattice need not be functional.

The set S_H of partial operators admits also an immediate description. Theorem 4.12 of [13] introduces, for any $B \in \mathcal{S}(H)$, a subset

$$L_B := \{P: P \leq P_B \text{ and } BP = PB\}$$

of $\mathcal{P}(H)$. For example, if $A \preceq B$, then $P_A \in L_B$. It is shown in the proof of the theorem that the mapping $\phi: A \mapsto P_A$ is an order isomorphism of the initial segment $[O, B]$ of $\mathcal{S}(H)$ onto L_B . Therefore,

$$L_B = \{P: P = P_A \text{ and } A \preceq B \text{ for some } A \in \mathcal{S}(H)\} = \{P_A: A \preceq B\}.$$

It is now easily seen that $S_H = \{B|_{\overline{\text{ran}} C}: P_C \in L_B\}$. Indeed, $B|_{\overline{\text{ran}} C}$ belongs to S_H , by definition, if and only if $B|_{\overline{\text{ran}} C} = A|_{\overline{\text{ran}} A}$ for some $A \in \mathcal{S}(H)$, i.e. if and only if $\overline{\text{ran}} C = \overline{\text{ran}} A$ and $B|_{\overline{\text{ran}} A} = A|_{\overline{\text{ran}} A}$, i.e., if and only if $P_C = P_A$ and $A \preceq B$.

It is also demonstrated in the proof of the mentioned theorem that

$$\text{if } P \leq P_B, \text{ then } P = P_B P \text{ and, further, } BP \preceq B \quad (4)$$

whenever $BP \in \mathcal{S}(H)$ i.e., whenever $BP = PB$. It follows that the inverse of the isomorphism ϕ takes a projection P from L_B into the operator $BP \in [O, B]$. Therefore, $[O, B] = \{BP : P \in L_B\}$.

4. MORE ON THE ORDERING OF $\mathcal{S}(H)$

In this section we discuss in more detail existence of joins and meets in the poset $\mathcal{S}(H)$.

It is noticed in [13, Theorem 4.12] that every poset L_B is a lattice with respect to \leq . According to this theorem, the initial segment $[O, B]$ of $\mathcal{S}(H)$ is order isomorphic to L_B ; thus, the segment itself is a lattice. Corollary 4.13 to this theorem then asserts that, in $\mathcal{S}(H)$, every pair of elements bounded above has a meet and a join. Of course, the meet of two elements in an initial segment of a poset is also their meet in the poset itself (and conversely). However, this may be not the case with joins; this fact seems to be overlooked in [13] when drawing the corollary. The subsequent theorem confirms that the corollary is nevertheless correct, and provides explicit descriptions of the corresponding partial operations (Corollary 6).

Recall that the commutant of an operator B (i.e., the set of all bounded operators commuting with B) is a von Neumann algebra. The lattice L_B , being an initial segment in the complete orthomodular lattice of all projections of this algebra, is therefore complete. This observation allows us to obtain a simple direct proof of existence of meets and joins in $\mathcal{S}(H)$ for all bounded sets of operators.

Theorem 4. *Suppose that $\mathcal{T} \subseteq \mathcal{S}(H)$, $\mathcal{T} \neq \emptyset$, and that $B \in \mathcal{S}(H)$ is an upper bound of \mathcal{T} . Then*

- (a) $B(\bigwedge(P_A : A \in \mathcal{T}))$ is the greatest lower bound of \mathcal{T} ,
- (b) $B(\bigvee(P_A : A \in \mathcal{T}))$ is the least upper bound of \mathcal{T} .

Proof. Assume that $A \preceq B$, i.e., $A = BP_A$, for all $A \in \mathcal{T}$. Then all projections P_A belong to the complete lattice L_B . This implies that the symbolic expressions in (a) and (b) present operators from $\mathcal{S}(H)$.

(a) Let $P_{\mathcal{T}}$ stand for the meet (in $\mathcal{P}(H)$) of all projections P_A with $A \in \mathcal{T}$. For every $A \in \mathcal{T}$, we have $BP_{\mathcal{T}} = BP_AP_{\mathcal{T}} = AP_{\mathcal{T}} = AP_{BP_{\mathcal{T}}}$ (by (4), as $P_{\mathcal{T}} \leq P_B$), i.e., $BP_{\mathcal{T}} \preceq A$. Hence, $BP_{\mathcal{T}}$ is a lower bound of \mathcal{T} . Further, if D is one more lower bound, then $D \preceq B$, $P_D \leq P_{\mathcal{T}}$ (as $P_D \leq P_A$ for all $A \in \mathcal{T}$), and $BP_{\mathcal{T}}P_D = BP_D = D$. Thus, $D \preceq BP_{\mathcal{T}}$. Therefore, $BP_{\mathcal{T}}$ is the greatest lower bound of \mathcal{T} .

(b) Let $P^{\mathcal{T}}$ stand for the join (in $\mathcal{P}(H)$) of all projections P_A with $A \in \mathcal{T}$. For every $A \in \mathcal{T}$, we have $A = BP_A = BP^{\mathcal{T}}P_A$, i.e., $A \preceq BP^{\mathcal{T}}$. Hence, $BP^{\mathcal{T}}$ is an upper bound of \mathcal{T} ; moreover, $BP^{\mathcal{T}} \preceq B$, for $P^{\mathcal{T}}$ belongs to the complete lattice L_B containing all P_A . Further, if D is one more upper bound, then likewise $DP_A = A$ for all $A \in \mathcal{T}$ and $DP^{\mathcal{T}} \preceq D$.

Now, $(B - D)P_A = 0$ and, consequently, $P_{B-D}P_A = 0$ for every $A \in \mathcal{T}$. Then in $\mathcal{P}(H)$ also $P_A \leq I - P_{B-D}$ for every A . Hence, $P^{\mathcal{T}} \leq I - P_{B-D}$, $P_{B-D}P^{\mathcal{T}} = 0$ and, finally, $(B - D)P^{\mathcal{T}} = 0$. Therefore, $BP^{\mathcal{T}} = DP^{\mathcal{T}} \preceq D$, and $BP^{\mathcal{T}}$ is the least upper bound of \mathcal{T} . \square

Notice that $\sup \mathcal{T} = O$ if $\mathcal{T} = \emptyset$. Therefore, item (b) of the theorem immediately implies the result of [19] that the poset $\mathcal{S}(H)$ is actually bounded complete (see Example 3 above), every initial segment of $\mathcal{S}(H)$ is a complete sublattice of $\mathcal{S}(H)$, and the embedding $P \mapsto BP$ of L_B into $\mathcal{S}(H)$ mentioned at the end of the previous section preserves all meets and joins.

As noted in the previous section, the ordering \preceq agrees on $\mathcal{P}(H)$ with the standard ordering \leq of projections. In [13], it is implicitly assumed without proof that then the meet (join) of two projection operators in $\mathcal{P}(H)$ is also their meet (resp., join) in the more extensive poset $\mathcal{S}(H)$. It follows from the theorem that this is indeed the case.

Corollary 5. $\mathcal{P}(H)$ is a complete sublattice of $\mathcal{S}(H)$.

Proof. Consider that $\mathcal{T} \subseteq \mathcal{P}(H)$ (then $P_A = A$ for every $A \in \mathcal{T}$) and $B = I$. \square

We write out the following significant particular case of the above theorem.

Corollary 6. If $A, B, C \in \mathcal{S}(H)$ and $A, B \preceq C$, then $A \wedge B$ and $A \vee B$ exist, and $A \wedge B = C(P_A \wedge P_B)$, $A \vee B = C(P_A \vee P_B)$.

By help of (4), we also conclude that

$$P_{A \wedge B} = P_A \wedge P_B, \quad P_{A \vee B} = P_A \vee P_B \quad (5)$$

whenever the pair A, B is bounded. Consequently, $\overline{\text{ran}}(A \vee B) = \overline{\text{ran}} A \sqcup \overline{\text{ran}} B$ and $\overline{\text{ran}}(A \wedge B) = \overline{\text{ran}} A \cap \overline{\text{ran}} B$ in this case. Examples 1 and 2 suggest that generally $\overline{\text{ran}}(A \wedge B) \subseteq \overline{\text{ran}} A \cap \overline{\text{ran}} B$.

The bounded completeness of $\mathcal{S}(H)$ implies that item (a) of Theorem 4 may be strengthened: every nonempty subset of $\mathcal{S}(H)$ has the greatest lower bound. In particular, the set of all lower bounds of operators $A, B \in \mathcal{S}(H)$ has the join, which is also its maximum element, i.e., $A \wedge B$. Observe that $\{P_C: C \preceq A \text{ and } C \preceq B\} = L_{A \wedge B}$, so that $P_{A \wedge B}$ belongs to this set and is the greatest element in it. Therefore, we come to the following description of meet of observables in a general case.

Corollary 7. For every pair of observables A and B , their meet exists and

$$A \wedge B = B(\max\{P_C: C \preceq A \text{ and } C \preceq B\}).$$

This particular result admits also a simple direct proof (cf. Proposition 3.2 in [1]). Let us consider the following subset of $\mathcal{P}(H)$:

$$L_{A,B} := \{P: P \leq P_A, P \leq P_B, AP = PA, BP = PB \text{ and } P \perp P_{A-B}\}.$$

If $C \preceq A, B$, then $AP_C = C = BP_C$, $(A - B)P_C = O$ and $P_C \perp P_{A-B}$. Moreover, then also $P_C \in L_A, L_B$. Therefore, $P_C \in L_{A,B}$. On the other hand, if $P \in L_{A,B}$, then $P \perp P_{A-B}$ and $P \perp (A - B)$, i.e., $(A - B)P = O$ and $AP = BP =: C$. As also $P \in L_A$ and $P \in L_B$, we conclude by (4) that $C \preceq A, B$. Therefore, $L_{A,B} = \{P: P = P_C \text{ and } C \preceq A, B \text{ for some } C \in \mathcal{S}(H)\}$ and, finally,

$$L_{A,B} = \{P_C: C \preceq A \text{ and } C \preceq B\} \subseteq L_A \cap L_B.$$

The definition of $L_{A,B}$ can be rewritten in the form

$$L_{A,B} = \{P \in \{A, B\}': P \leq P_A \wedge P_B \wedge (I - P_{A-B})\},$$

where $\{A, B\}'$ is the commutant of $\{A, B\}$ and I is the identity operator. Therefore, $L_{A,B}$ is a bounded subset of the complete lattice of all projections of the von Neumann algebra $\{A, B\}'$ and, hence, has the join P^* . Evidently, P^* is even the maximum element of $L_{A,B}$. It is easily seen that then the operator BP^* is the g.l.b. of A and B in $\mathcal{S}(H)$. Indeed, since $P^* \in L_{A,B}$, it follows that $P^* \in L_A, L_B$ and $P^* = P_C$ for some C with $C \preceq A, B$. So, $C = AP^* = BP^*$. Suppose that D is one more lower bound of A and B . As $P_D \in L_{A,B}$, it follows that $P_D \leq P^* = P_C$. On

the other hand, $P_D \in L_A, L_B$. Recall that the transfer $\phi: E \mapsto P_E$ (Section 3) is an order isomorphism of $[O, B]$ onto L_B ; therefore $D \preceq C$, and $C = A \wedge B$.

Of course, AP^* also is the g.l.b. of A and B . We already know that actually $P^* = P_{A \wedge B}$.

5. QUASI-ORTHOMODULAR NEARSEMILATTICES

In [14], M.F. Janowitz introduced the notion of a generalized orthomodular lattice, which was defined as a lattice equipped with an appropriate orthogonality relation. We take up the idea and consider in this section nearsemilattices with orthogonality.

A binary relation \perp on a poset A with the least element 0 is said to be an *orthogonality*, if it satisfies the conditions

- (\perp_1): if $x \perp y$, then $y \perp x$,
- (\perp_2): if $x \leq y$ and $y \perp z$, then $x \perp z$,
- (\perp_3): $x \perp 0$.

For example, if $\bar{}$ is an m-complementation on A and a relation \perp is defined by $x \perp y$ iff $y \leq x^\perp$, then \perp is an orthogonality on A . We say that it is *induced* by the m-complementation $\bar{}$. Evidently, the induced orthogonality is *additive* in the sense that

- (\perp_4): if $x \perp y$, $x \perp z$ and $y \circ z$, then $x \perp y \vee z$.

Definition 1. Suppose that $(A, \vee, 0)$ is a nearsemilattice and that \perp is an orthogonality on it. The algebraic system $(A, \vee, \perp, 0)$ is called a *quasi-orthomodular nearsemilattice* if the following additional conditions are fulfilled:

- (\perp_5): if $x \perp y$, then $x \circ y$,
- (\perp_6): if $x \leq y$, then $y = x \vee z$ for some z with $x \perp z$,
- (\perp_7): if $x \perp y$, $x \perp z$ and $y \leq x \vee z$, then $y \leq z$.

In a quasi-orthomodular nearsemilattice, the following cancellation law holds:

- (\perp_8): if $x \perp y$, $x \perp z$ and $x \vee y = x \vee z$, then $y = z$.

In particular,

- (\perp_9): if $x \perp x$, then $x = 0$.

Also, (\perp_4) together with (\perp_5) implies that

- (\perp_{10}): every finite set of mutually orthogonal elements has a join

(recall that in a nearsemilattice $x \circ y$ iff x and y have a common upper bound).

Example 1 (continuation). In a functional nearsemilattice, put $\varphi \perp \psi$ if $\text{dom } \varphi \cap \text{dom } \psi = \emptyset$. Then $(\mathcal{PF}(I, V), \cup, \perp, \lambda)$ is a quasi-orthomodular nearsemilattice with an additive orthogonality. The existential condition (\perp_6) may fail in an arbitrary closed functional nearsemilattice. The other existential condition (\perp_5) is ensured by closedness and may fail in an arbitrary functional nearsemilattice.

Example 2 (continuation). The relation \perp on $\mathcal{M}(\mathcal{A})$ introduced in the Section 3 is an orthogonality, and $(\mathcal{M}(\mathcal{A}), \gamma, \perp, 0)$ is a quasi-orthomodular nearsemilattice. The orthogonality satisfies also (\perp_4) (Theorem 3.2(b) in [19]). Taking into account that (\perp_6) follows directly from the definition of \preceq , these facts can be derived from the previous example, because the isomorphism of $\mathcal{M}(\mathcal{A})$ onto the closed functional nearsemilattice $M(A)$, which was described in Section 3, preserves orthogonality.

Example 3 (continuation). Likewise, $(\mathcal{S}(H), \gamma, \perp, O)$ is a quasi-orthomodular nearsemilattice for \perp defined as in Section 3. Indeed, (i) if $AB = O$, then $BA = O$ [13, Lemma 4.1]; (ii) if $A \preceq B$ and $BC = O$, then $A = BP_A = P_AB$ and $AC = O$; (iii) $AO = O$; thus, \perp is an orthogonality. Further, (v) if $AB = O$, then $A \preceq A + B$ and $B \preceq A + B$ by the definition of \preceq , and $A \circ B$ (as $\mathcal{S}(H)$ is a nearsemilattice) [in addition, (v') $A + B = A \vee B$; see the proof of Theorem 4.3 in [19]]; (vi) if $A \preceq B$, then, by the definition and (v'), $B = A + C = A \vee C$ for some C with $A \perp C$; (vii) if $AB = O$, $AC = O$ and $B \preceq A \vee C$, then $A + C = B + D$ for some D with $DB = O$, $(D - A)B = O$ and, further, $C = B + (D - A)$, i.e., $B \preceq C$.

It was proved in Theorem 4.3 of [19] that a weakened version of (\perp_4) holds in $\mathcal{S}(H)$. The orthogonality on $\mathcal{S}(H)$ is in fact additive. Indeed, suppose that $D \perp A$, $D \perp B$ and $A \circ B$. Then also $P_D \perp P_A$ and $P_D \perp P_B$. As the lattice $\mathcal{P}(H)$ is orthomodular, it follows that $P_D \perp P_A \vee P_B$. On the other hand, $P_A \vee P_B = P_{A \vee B}$ —see (5). Therefore, $P_D \perp P_{A \vee B}$ and $D \perp A \vee B$.

The isomorphism of $\mathcal{S}(H)$ onto the nearsemilattice S_H described in the Section 3 preserves orthogonality; so we may conclude that S_H is quasi-orthomodular.

Theorem 4.12 of [13] asserts that every initial segment of $\mathcal{S}(H)$ is isomorphic to an orthomodular lattice. We are now going to generalize this structure theorem to arbitrary quasi-orthomodular nearsemilattices. Recall that an orthocomplementation, or an *o-complementation*, for short, on a bounded poset $(P, \leq, 0, 1)$ is an m-complementation $-$ such that $1 = x \vee x^-$ (equivalently, $0 = x \wedge x^-$) for all $x \in P$. Observe that then the induced orthogonality satisfies (\perp_9) . An o-complemented poset is *orthomodular* if this orthogonality satisfies also (\perp_5) and (\perp_6) ; conditions (\perp_7) and (\perp_4) are fulfilled in every such a poset. An *orthomodular lattice* is a lattice-ordered orthomodular poset.

Theorem 8. *Every initial segment of a quasi-orthomodular nearsemilattice A is an orthomodular lattice, where joins and meets agree with those existing in A .*

Proof. Due to (\perp_6) and (\perp_8) , there is, for every $x \leq p$, a unique element $y \in [0, p]$ such that $x \perp y$ and $x \vee y = p$; we denote this element by x_p^- .

The mapping $x \mapsto x_p^-$ is an m-complementation on $[0, p]$. By the definition, $(x_p^-)_p \perp x_p^-$ and $(x_p^-)_p \vee x_p^- = p$; hence $(x_p^-)_p = x$. If $x \leq y \leq p$, then $x \perp y_p^-$ by (\perp_2) and $y_p^- \leq x_p^-$ by (-7) (observe that $y_p^- \leq x \vee x_p^-$).

As $x \vee x_p^- = p$, the m-complementation $\bar{\cdot}_p$ is even an o-complementation in $[0, p]$, and the interval $[0, p]$ is an o-complemented poset. In virtue of (\perp_5) and (\perp_6) , the poset is orthomodular. Since an initial segment $[0, p]$ of a nearsemilattice is actually an upper semilattice by definition, the o-complementation $\bar{\cdot}_p$ turns it into a lattice. The final assertion is now trivial. \square

The following conclusion is immediate (recall that a distributive semilattice that happens to be a lattice is also distributive as a lattice).

Corollary 9. *In a distributive quasi-orthomodular nearsemilattice, every initial segment is a Boolean algebra.*

It is observed in [13] that $\mathcal{M}(A)$ is a generalized orthoalgebra, and [13, Theorem 4.2] asserts that so is also $\mathcal{S}(H)$. This result extends to arbitrary quasi-orthomodular nearsemilattices.

Definition 2. A *generalized orthoalgebra* is a system $(A, \oplus, 0)$, where \oplus is a partial binary operation and 0 is a nullary operation on A , satisfying the conditions (we write here, for arbitrary terms s and t , $s \perp t$ to mean that $s \oplus t$ is defined):

- (\oplus_1) : if $x \perp y$, then $y \perp x$ and $x \oplus y = y \oplus x$,
- (\oplus_2) : if $x \perp y$ and $x \oplus y \perp z$, then
 $y \perp z, x \perp y \oplus z$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
- (\oplus_3) : $x \perp 0$ and $x \oplus 0 = x$,
- (\oplus_4) : if $x \perp y, x \perp z$ and $x \oplus y = x \oplus z$, then $y = z$,
- (\oplus_5) : if $x \perp x$, then $x = 0$.

The relation \leq defined by

$$x \leq y \text{ if and only if } y = x \oplus z \text{ for some } z \text{ with } x \perp z, \quad (6)$$

is an order on a generalized orthoalgebra A and is called its *natural ordering*.

Theorem 10. Suppose that $(A, \vee, \perp, 0)$ is a quasi-orthomodular nearsemilattice and that \oplus is a binary operation on A defined by

$$x \oplus y = z \text{ if and only if } x \perp y \text{ and } z = x \vee y. \quad (7)$$

Then $(A, \oplus, 0)$ is a generalized orthoalgebra, and its natural ordering coincides with the nearsemilattice ordering of A .

Proof. In virtue of (\perp_5) , the axioms (\oplus_1) – (\oplus_5) are easy consequences of $(\vee_2) + (\perp_1)$, $(\vee_3) + (\perp_2)$, $(\vee_4) + (\perp_3)$, (\perp_8) and (\perp_9) , respectively. We only note in connection with (\oplus_2) that the supposition $x \oplus y \perp z$ implies that (i) $x, y \perp z$ (see (\perp_2)) and (ii) $x \vee y \perp z$, i.e., $x, y, z \leq p$ for some p . Then (in the orthomodular lattice $[0, p]$) $y, z, y \vee z \leq x_p^-$ and $x \perp y \vee z$, i.e., $x \perp y \oplus z$. The equivalence (6) follows from (\perp_6) . \square

Example 2 (continuation). The operation \oplus defined by $f \oplus g := f + g$ for $f \perp g$ turns $\mathcal{M}(\mathcal{A})$ into a generalized orthoalgebra [13]. See the item (i) in the proof of [19, Theorem 3.2] for (7).

Example 3 (continuation). If $A \perp B$, then put $A \oplus B := A + B$ [13]. Then $\mathcal{S}(H)$ is a generalized orthoalgebra [13, Theorem 4.2]. Evidently, $A \oplus B$ is an upper bound of A and B , and Corollary 4.5 in [13] says that it is actually a least upper bound. Thus, (7) also holds in $\mathcal{S}(H)$.

The theorem implies that an orthomodular nearsemilattice satisfying (\perp_4) is a generalized orthomodular poset (see [19] for an appropriate version of the latter notion).

We say that a quasi-orthomodular nearsemilattice has the *Riesz decomposition property* if it satisfies the condition

$$(\oplus_6): \text{ if } y \perp z \text{ and } x \leq y \oplus z, \text{ then } x = y' \oplus z' \text{ for some } y' \leq y \text{ and } y' \leq z,$$

where \oplus is the operation (7) (cf. [11] or [19, Sect. 2]). By (\perp_5) and (7), this property turns out to be equivalent to distributivity.

Theorem 11. A quasi-orthomodular nearsemilattice is distributive if and only if it has the Riesz decomposition property.

Proof. In a quasi-orthomodular nearsemilattice A , if $y \perp z$ and $x \leq y \oplus z$, then $x \leq y \vee z$ and, by distributivity, $x = y' \vee z'$ for some $y' \leq y$ and $z' \leq z$. But $y' \perp z'$ in virtue of (\perp_2) ; therefore, $x = y' \oplus z'$. Now suppose that A has the Riesz

decomposition property and that $y \perp z$ and $x \leq y \vee z$. By (\perp_6) , $y \vee z = y \vee z_0$ for some z_0 with $z_0 \perp y$. As $z \leq y \vee z_0$, there are $y_1 \leq y$ and $z_1 \leq z_0$ such that $y_1 \perp z_1$ and $z = y_1 \vee z_1$. Then $y \vee z = y \vee z_1$, $z_1 \leq z$ and, in virtue of (\perp_2) , $y \perp z_1$. It follows that $x = y' \vee z'$ for some $y' \leq y$ and $z' \leq z_1 \leq z$. \square

6. SKEW MEETS ON QUASI-ORTHOMODULAR NEARSEMI-LATTICES

We concentrate in this section on quasi-orthomodular nearsemilattices that admit a non-commutative meet-like operation, and demonstrate that the nearsemilattices in Examples 1–3 belong to this type of nearsemilattices.

Let A be a quasi-orthomodular nearsemilattice, and assume that orthogonality in it is additive. For $x, y \in A$, we write $x \sqsubset y$ to mean that, for every $z \in A$, $z \perp y$ only if $z \perp x$. We say that x is *overridden* by y , if $x \sqsubset y$. The overriding relation \sqsubset has the following properties:

- (\sqsubset_1): \sqsubset is reflexive and transitive,
- (\sqsubset_2): if $x \leq y$, then $x \sqsubset y$.
- (\sqsubset_3): if $x \sqsubset y$ and $x \perp y$, then $x \leq y$,
- (\sqsubset_4): if $x \sqsubset z$, $y \sqsubset z$ and $x \perp y$, then $x \vee y \sqsubset z$.

Only (\sqsubset_3) requires some comment. By (\perp_6) , $x \vee y = x_0 \vee y$ for some x_0 with $x_0 \perp y$. Since $x \sqsubset y$, then $x_0 \perp x$ and, further, $x_0 \perp x \vee y$ (see (\perp_4)). Now, $x_0 = 0$ by (\perp_2) and (\perp_9) .

Therefore, \sqsubset is a preorder. We denote by \parallel the equivalence relation on A induced by it: $x \parallel y$ iff $x \sqsubset y$ and $y \sqsubset x$. Abstract overriding relations satisfying (\sqsubset_1)–(\sqsubset_4) and one more condition

- (\sqsubset_5): if $x \sqsubset y$, then $x \parallel x' \leq y$ for some x' ,

were introduced in [4, Definition 2.2]. Observe that the element x' in the right side of (\sqsubset_5) is uniquely defined; in fact, $x' = \max\{u : u \sqsubset x \text{ and } u \leq y\}$. Indeed, $x' \sqsubset x$ and $x' \leq y$ by the definition of \parallel ; on the other hand, if $u \sqsubset x$ and $u \leq y$, then $u \sqsubset x'$ and $u \perp x'$, whence $u \leq x'$ by (\sqsubset_3).

If the element

$$x \overleftarrow{\wedge} y := \max\{u : u \sqsubset x \text{ and } u \leq y\}$$

exists for some x and y , we call it a (right-handed) *skew meet* of x and y .

Example 1 (continuation). In $\mathcal{PF}(I, V)$, $\varphi \sqsubset \psi$ if and only if $\text{dom } \varphi \subseteq \text{dom } \psi$. This overriding relation on $\mathcal{PF}(I, V)$ satisfies (\sqsubset_5): if $\varphi \sqsubset \psi$ and $\varphi' := \psi|_{\text{dom } \varphi}$, then $\varphi \parallel \varphi' \subseteq \psi$. However, (\sqsubset_5) may fail in an arbitrary functional nearsemilattice (even if the latter is closed). More generally, the skew intersection $\overleftarrow{\cap}$ on $\mathcal{PF}(I, V)$ is totally defined and is given by $\varphi \overleftarrow{\cap} \psi = \psi|_{\text{dom}(\varphi \cap \psi)}$ (see [5]). Observe that $\varphi \subseteq \psi$ iff $\varphi \overleftarrow{\cap} \psi = \varphi$, $\varphi \perp \psi$ iff $\varphi \overleftarrow{\cap} \psi = \lambda$ and $\varphi \sqsubset \psi$ iff $\psi \overleftarrow{\cap} \varphi = \varphi$.

Example 2 (continuation). In $\mathcal{M}(\mathcal{A})$, $f \sqsubset g$ if and only if $\text{supp } f \subseteq \text{supp } g$. In connection with (\sqsubset_5), observe that if $f \sqsubset g$, then the function f' which agrees with g on $\text{supp } f$ and vanishes outside $\text{supp } f$ belongs to $\mathcal{M}(\mathcal{A})$. More generally, the skew meet $\overleftarrow{\wedge}$ on $\mathcal{M}(\mathcal{A})$ is a total operation and is given by $f \overleftarrow{\wedge} g = g\chi_{(\text{supp } f \cap \text{supp } g)}$.

Example 3 (continuation). In $\mathcal{S}(H)$, $A \sqsubset B$ if and only if $\overline{\text{ran}} A \subseteq \overline{\text{ran}} B$. Indeed, suppose that $C \perp A$ whenever $C \perp B$, and choose $x \in \overline{\text{ran}} A$. Take for C the projection onto the closed subspace $[x]$ spanned by x . Then $C \not\perp A$ and, consequently, $C \not\perp B$. Hence, $[x] \subseteq \overline{\text{ran}} B$, and $x \in \overline{\text{ran}} B$. Conversely, if $\overline{\text{ran}} A \subseteq \overline{\text{ran}} B$ and $C \perp B$, then $\overline{\text{ran}} C \cap \overline{\text{ran}} A \subseteq \overline{\text{ran}} C \cap \overline{\text{ran}} B$ and $C \perp A$.

Equivalently, $A \sqsubset B$ if and only if $P_A \leq P_B$. The axiom (\sqsubset_5) means that if $P_A \leq P_B$, then $P_A = P_{A'}$ and $A' \preceq B$ for some $A' \in \mathcal{S}(H)$ (equivalently, $P_A \in L_B$, or $B \mid \overline{\text{ran}} A \in S_H$). A natural candidate for A' is the operator BP_A ; we, however cannot prove that it is self-adjoint, i.e., that B and P_A commute.

However, the skew meet operation $\overleftarrow{\wedge}$ in $\mathcal{S}(H)$ is total. The bounded subset $L_B^A := \{P: P \leq P_A, P_A \leq P_B \text{ and } BP = PB\}$ of the complete lattice L_B always has the greatest element P^* . On the other hand,

$$L_B^A = \{P_C \in L_B: P_C \leq P_A\} = \{P_C: C \sqsubset A \text{ and } C \preceq B\}.$$

Now, a reasoning similar to that at the end of Section 4 demonstrates that BP^* is the skew meet of A and B .

We end with a theorem describing characteristic properties of the skew meet operation $\overleftarrow{\wedge}$.

Theorem 12. *Suppose that A is a quasi-orthomodular nearsemilattice with an additive orthogonality and that all skew meets $x \overleftarrow{\wedge} y$ in A exist. Then the algebra $(A, \overleftarrow{\wedge})$ is an idempotent semigroup, and the following conditions are fulfilled:*

$$x \overleftarrow{\wedge} y \leq y, \quad x \overleftarrow{\wedge} y \sqsubset x, \quad (8)$$

$$x \leq y \text{ iff } x \overleftarrow{\wedge} y = x, \quad y \sqsubset x \text{ iff } x \overleftarrow{\wedge} y = y. \quad (9)$$

Moreover, $\overleftarrow{\wedge}$ is commutative in every initial segment of A .

Proof. Evidently, the operation $\overleftarrow{\wedge}$ is idempotent, and both inequalities (8) are trivial. To prove that the operation is associative, observe that

$$\begin{aligned} (x \overleftarrow{\wedge} y) \overleftarrow{\wedge} z &= \max(v: v \sqsubset \max(u: u \sqsubset x \text{ and } u \leq y) \text{ and } v \leq z) \\ &= \max(v: v \sqsubset u \sqsubset x \text{ and } u \leq y \text{ for some } u, \text{ and } v \leq z) \end{aligned} \quad (10)$$

and

$$\begin{aligned} x \overleftarrow{\wedge} (y \overleftarrow{\wedge} z) &= \max(v: v \sqsubset x \text{ and } v \leq \max(u: u \sqsubset y \text{ and } u \leq z)) \\ &= \max(v: v \sqsubset x, \text{ and } v \leq u \leq z \text{ and } u \sqsubset y \text{ for some } u) \\ &= \max(v: v \sqsubset x, v \leq z \text{ and } v \sqsubset y). \end{aligned} \quad (11)$$

Now notice that if an element v satisfies, for some u , the conditions $v \sqsubset u \sqsubset x$, $u \leq y$ and $v \leq z$ from (10), then it satisfies also the conditions $v \sqsubset x$, $v \leq z$ and $v \sqsubset y$ from (11). Conversely, if the latter triple of conditions is satisfied, then also the former one is satisfied with $u = x \overleftarrow{\wedge} y$. We only note that if $v \sqsubset x, y$, then there is v' such that $v \parallel v' \leq y$ (see (\sqsubset_5)), and, further, $v' \leq u$; it follows that $v \sqsubset u$.

Therefore, the maxima in (10) and (11) are equal. Next, if $x \leq y$ and $u \leq y$, then $x \sqcup u$ and, if also $u \sqsubset x$, then $u \leq x$. Thus, $x \overleftarrow{\wedge} y = x$. The converse is evident by (8), and this proves the first identity in (9). Further, if $y \sqsubset x$, then $y \leq x \overleftarrow{\wedge} y$, and the converse again comes from (8); this proves the other identity.

At last, it follows from (\sqsubset_3) by virtue of (\perp_2) that $x \overleftarrow{\wedge} y = x \wedge y = y \overleftarrow{\wedge} x$ when $x \sqcup y$. \square

The theorem shows that the algebra $(A, \vee, \overleftarrow{\wedge}, 0)$ is a right normal skew nearlattice in the sense of [5, 6]. Thus, in particular, $\mathcal{S}(H)$ is such an algebra.

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